TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 172, October 1972

EXTENSIONS OF HOLOMORPHIC MAPS

BY

PETER KIERNAN

ABSTRACT. Several generalizations of the big Picard theorem are obtained. We consider holomorphic maps f from $X-A$ into $M \subset Y$. Under various assumptions on X, A, and M we show that f can be extended to a holomorphic or meromorphic map of X into Y.

1. **Introduction.** The big Picard theorem says that any holomorphic map f from the punctured disk D^* into the Riemann sphere $P_1(\mathbf{C})$ which omits three points can be extended to a holomorphic map $f: D \to P_1(\mathbf{C})$. Kobayashi [3] has obtained the following generalization of this. Let A be a closed submanifold of the complex manifold X and let M be a hyperbolically imbedded(1) subspace (definition below) of the complex space Y. Then any holomorphic map $f: X - A \to M$ can be extended to a holomorphic map $f: X \to Y$. The purpose of this paper is to consider further generalizations of this result. In particular, it will be shown that if A has singularities, then in general f can only be extended to a meromorphic map. However, if the singularities of A are normal crossings, then f can be extended to a holomorphic map into Y.

2. **Definitions.** Our main tool for investigating the extension of holomorphic maps will be the Kobayashi pseudo-distance d_M which is associated with the complex space M. For completeness, we shall give the definition and a summary of some of the main properties of d_M which we use. The reader can consult [3] for further details.

Let p and q be points in the complex space M. By a *chain* α from p to q, we mean a sequence of points $p = p_0, p_1, \cdots, p_k = q$ in M, points $a_1, \cdots, a_k$ in the unit disk $D = \{z \in \mathbf{C} \mid |z| < 1\}$ and holomorphic maps $f_1, \cdots, f_k$ of D into M with $f_i(0) = p_{i-1}$ and $f_i(a_i) = p_i$. The length $|\alpha|$ of α is defined by

$$|\alpha| = \sum_{i=1}^{k} d(0, a_i) = \sum_{i=1}^{k} \log \frac{1 + |a_i|}{1 - |a_i|}$$

Received by the editors February 1, 1971 and, in revised form, October 20, 1971.
AMS (MOS) subject classifications (1970). Primary 32H25, 32H20.
Key words and phrases. Big Picard theorem, hyperbolic, Kobayashi pseudo-distance, Satake compactification, symmetric bounded domain.

(1) Several characterizations of hyperbolically imbedded spaces and another application of Theorem 2 are given in a recent paper by this author. The paper is entitled *Hyperbolically imbedded spaces and the big Picard theorem* and has been submitted to Mathematische Annalen.

Copyright © 1973, American Mathematical Society

where d is the Poincaré-Bergman distance on D. It is given by the metric

$$ds^2 = \frac{1}{(1-|z|^2)^2} dz d\bar{z}.$$

We set $d_M(p, q) = \inf_{\alpha \in A} |\alpha|$, where A is the set of all chains from p to q. It is easy to see that d_M is a pseudo-distance on M. If d_M is a (complete) metric, we say that M is *(complete) hyperbolic.*

Some of the properties of this pseudo-distance which shall be useful to us are

(1) d_D is the Poincaré-Bergman distance on the unit disk D.

(2) Let $D^* = \{z \in \mathbf{C} \mid 0 < |z| < 1\}$. Then d_{D^*} is a complete metric corresponding to the hermitian metric

$$ds^2 = \frac{1}{|z|^2(\log|z|^2)^2} dz d\bar{z}.$$

Thus D^* is complete hyperbolic. However, if $\gamma_r = \{z = re^{it} | 0 \le t \le 2\pi\}$, then the diameter of γ_r converges to 0 as r converges to 0.

(3) If $f: M \to N$ is holomorphic, then f is distance decreasing. That is, if $p, q \in M$ then $d_N(f(p), f(q)) \le d_M(p, q)$.

(4) M is (complete) hyperbolic iff the universal covering space of M is (complete) hyperbolic.

Let Y be a complex space and let M be a relatively compact, hyperbolic complex subspace of Y. We say that M is *hyperbolically imbedded* in Y if either of the following (equivalent) conditions are satisfied:

(i) If p and q are boundary points of M and if $\{p_n\}$ and $\{q_n\}$ are sequences in M such that $p_n \to p$, $q_n \to q$, and $d_M(p_n, q_n) \to 0$, then $p = q$.

(ii) If p is a boundary point of M and U is a neighborhood of p in Y, there exists a neighborhood V of p in Y such that $\bar{V} \subset U$ and the distance between $M \cap (Y - U)$ and $M \cap V$ with respect to d_M is positive.

Intuitively, a relatively compact subspace M is hyperbolically imbedded in Y if M is hyperbolic and if the "distance" between any two distinct boundary points is greater than zero.

The following are some examples of hyperbolically imbedded spaces.

(5) Let M be a compact hyperbolic manifold. Then M is hyperbolically imbedded in $Y = M$.

(6) Let $Y = P_1(\mathbf{C})$ and $M = P_1(\mathbf{C}) - \{\text{three points}\}$. M is covered by the unit disk and therefore it is complete hyperbolic. Since M has only isolated boundary points, M is hyperbolically imbedded in Y.

(7) If M_1 and M_2 are hyperbolically imbedded in Y_1 and Y_2 respectively, then $M_1 \times M_2$ is hyperbolically imbedded in $Y_1 \times Y_2$. To see this let $\pi_j: M_1 \times M_2 \to M_j$ be the projection onto M_j and let $p = (p', p'')$, $q = (q', q'')$, $\{p_n = (p_n', p_n'')\}$, and $\{q_n = (q_n', q_n'')\}$ be as in (i) above. If $d_{M_1 \times M_2}(p_n, q_n) \to 0$, then $d_{M_1}(p_n', q_n') \to 0$ and $d_{M_2}(p_n'', q_n'') \to 0$ since π_1 and π_2 are distance de-

creasing. Thus $p' = q'$ and $p'' = q''$ and therefore $M_1 \times M_2$ is hyperbolically imbedded in $Y_1 \times Y_2$.

(8) Let $P_2(\mathbf{C})$ be the 2-dimensional complex projective space and let Q be a complete quadrilateral. That is, Q is the union of the six projective lines which pass through a set of four points in general position. Then $M = P_2(\mathbf{C}) - Q$ is hyperbolically imbedded in $P_2(\mathbf{C})$; see [3, p. 92].

Finally, we recall the definition of meromorphic map given by Remmert in [6]. A meromorphic mapping f from a complex space X into a complex space Y is a correspondence satisfying

(i) For each point $x \in X$, $f(x)$ is a nonempty compact subset of Y.

(ii) The graph $\Gamma_f = \{(x, y) \in X \times Y | y \in f(x)\}$ of f is a connected complex subspace of $X \times Y$ with $\dim \Gamma_f = \dim X$.

(iii) There exists a dense subset X' of X such that $f(x)$ is a single point for each x in X'. Remmert has shown that the codimension of the set $X - X'$ is ≥ 2. Thus if X is a normal space and $\dim X = 1$, then f is actually holomorphic.

3. Extension theorems.

The following theorem is the basis for our extension theorems. The proof is essentially the same as Mrs. Kwack's proof of Theorem 3 in [5], although our statement is slightly more general. The winding number argument used below is due to Grauert and Reckziegel.

Theorem 1. *Let M be hyperbolically imbedded in Y and let $f_k\colon D^* \to M$ be a sequence of holomorphic maps where $D^* = \{z \in \mathbf{C} | 0 < |z| < 1\}$. Let $\{z_k\}$ and $\{z_k'\}$ be sequences in D^* converging to 0 and such that $f_k(z_k') \to q \in Y$. Then*

(i) *$f_k(z_k) \to q$.*

(ii) *Any holomorphic map $f\colon D^* \to M$ extends to a holomorphic map $f\colon D \to Y$.*

(iii) *$f_k(0) \to q$.*

Proof. (i) Assume that $f_k(z_k) \to p \neq q$ and that $|z_k| \leq |z_k'|$. Choose a coordinate neighborhood U of p such that U is an analytic subset of $W = \{(w_1, \cdots, w_n) \in \mathbf{C}^n | |w_1|^2 + \cdots + |w_n|^2 < 2\}$, $q \notin U$, and $p = (0, \cdots, 0)$. Let $\rho_k(t) = z_k e^{it}$ for $0 \leq t \leq 2\pi$. Then the diameters of the sets ρ_k with respect to d_{D^*} are converging to 0. Since M is hyperbolically imbedded in Y and since f_k is distance decreasing, this implies $f_k(\rho_k)$ is converging to p. Thus for all sufficiently large k, there exists an annulus R_k' centered at 0 such that $\rho_k \subset R_k'$ and $f_k(R_k') \subset B = \{(w_1, \cdots, w_n)| |w_1|^2 + \cdots + |w_n|^2 < 1\}$. Let R_k be the largest such annulus. We can assume that either R_k is not a punctured disk for any k or that R_k is a punctured disk for all k. We consider the former case first. Since $f_k(z_k') \to q \notin U$, there exist a_k and b_k in $\overline{R}_k$ with $|a_k| < |b_k|$ and $|f_k(a_k)| = |f_k(b_k)| = 1$. By taking a subsequence and relabelling, we may assume that $f_k(a_k) \to q' \in \overline{B}$ and $f_k(b_k) \to q'' \in \overline{B}$. Let σ_k and τ_k be the curves defined by $\sigma_k(t) = a_k e^{it}$ and $\tau_k(t) = b_k e^{it}$ for $0 \leq t \leq 2\pi$. By the same reasoning used for ρ_k, we see that $f_k(\sigma_k) \to q'$ and $f_k(\tau_k) \to q''$.

Let $f_k = (f_k^1, \cdots, f_k^n)$ in $f_k^{-1}(U)$. By rotating the coordinate ball W if necessary, we see that the winding number of the curves $f_k^1(\sigma_k)$ and $f_k^1(\tau_k)$ around the point $f_k^1(z_k)$ is 0 for all k sufficiently large. By Cauchy's theorem, we have

$$\int_{\sigma_k} \frac{f_k^{1\prime}(z)}{f_k^1(z) - f_k^1(z_k)}\, dz = \int_{f_k^1(\sigma_k)} \frac{dw_1}{w_1 - f_k^1(z_k)} = 0$$

and

$$\int_{\tau_k} \frac{f_k^{1\prime}(z)}{f_k^1(z) - f_k^1(z_k)}\, dz = \int_{f_k^1(\tau_k)} \frac{dw_1}{w_1 - f_k^1(z_k)} = 0.$$

Thus

$$0 = \int_{\sigma_k} \frac{f_k^{1\prime}(z)}{f_k^1(z) - f_k^1(z_k)}\, dz - \int_{\tau_k} \frac{f_k^{1\prime}(z)}{f_k^1(z) - f_k^1(z_k)}\, dz = 2\pi i(P - N)$$

where N and P are the number of zeros and poles of the function $f_k^1(z) - f_k^1(z_k)$ on the annulus R_k. This is a contradiction since $N > 0$ and $P = 0$.

If R_k is a punctured disk for all k, then the boundary of $\overline{R}_k$ is τ_k where b_k and τ_k are as above. Since $f_k(R_k) \subset B$, f_k extends holomorphically to $\overline{R}_k$ with $f_k(0)$ in B. Setting $\sigma_k = \emptyset$, the argument used in the preceding paragraph leads to a contradiction. This proves (i).

(ii) Let $\{z_k'\}$ be any sequence in D^* such that $f(z_k') \to q \in Y$. Define $f(0) = q$. By (i), f is continuous. The Riemann extension theorem now implies that f is holomorphic at 0.

(iii) If $f_k(0)$ does not converge to q, then without loss of generality we may assume that $f_k(0) \to p \neq q$. Since each f_k is continuous, there exists a sequence $\{z_k\}$ in D^* such that $z_k \to 0$ and $f_k(z_k) \to p$. This contradicts (i). Q.E.D.

The following theorem was proven by Kwack in the case when M is compact and by Kobayashi in the case when A is nonsingular.

Theorem 2. *Let A be a closed complex subspace of a (nonsingular) complex manifold X. If the singularities of A are normal crossings and if M is hyperbolically imbedded in Y, then any holomorphic map $f: X - A \to M$ extends to a holomorphic map $f: X \to Y$.*

Proof. Since the singularities of A are normal crossings, we can assume that

$$X = D \times \cdots \times D = D^n \times D^l = \{(w_1, \cdots, w_n, t_1, \cdots, t_l) \mid |w_i| < 1 \text{ and } |t_j| < 1\}$$

and that

$$X - A = (D^*)^n \times D^l = \{(w_1, \cdots, w_n, t_1, \cdots, t_l) \mid 0 < |w_i| < 1 \text{ and } |t_j| < 1\}.$$

The proof is by induction on n.

Case 1. $X - A = D^*$. This is part (ii) of Theorem 1.

Case 2. Assume we can extend f when $X - A = (D^*)^n$. We show that this implies we can extend f if $X - A = (D^*)^n \times D^l$.

Let $w = (w_1, \cdots, w_n) \in D^n$ and $t = (t_1, \cdots, t_l) \in D^l$. We denote the point $(w_1, \cdots, w_n, t_1, \cdots, t_l) \in D^n \times D^l$ by (w, t). Let $f: (D^*)^n \times D^l \to M$ be holomorphic and define $f_t: (D^*)^n \to M$ by $f_t(w) = f(w, t)$. Since each map f_t extends to D^n, we can extend f to a map from $D^n \times D^l$ into Y by defining $f(w, t) = f_t(w)$. By the Riemann extension theorem, it suffices to show that the extended map f is continuous.

We assume that f is not continuous at some point, say $(w, 0)$ for simplicity, and then obtain a contradiction. If f is not continuous at $(w, 0)$, then there exists a sequence of points (w^k, t^k) in $(D^*)^n \times (D^*)^l$ with $(w^k, t^k) \to (w, 0)$ and such that the sequence $f(w^k, t^k) \to q \neq f(w, 0)$. Define $f_k: D^* \to M$ by $f_k(z) = f(w^k, zt^k/|t^k|)$. Let $z_k' = |t^k|$. Since $z_k' \to 0$ and $f_k(z_k') = f(w^k, t^k) \to q$, part (iii) of Theorem 1 implies that $f_k(0) = f(w^k, 0) \to q$. But f_t is continuous for each t and therefore $f(w^k, 0) \to f(w, 0) \neq q$. Thus the assumption that f is not continuous is false.

Case 3. Assume that f can be extended if $X - A = (D^*)^n \times D^l$. We show that this implies that f can be extended if $X - A = (D^*)^{n+1}$. By the inductive hypothesis, we can extend f to $D^{n+1} - \{(0, \cdots, 0)\}$. The map $g: D^* \to M$ defined by $g(z) = f(z, \cdots, z)$ extends to D. Define $f(0, \cdots, 0) = g(0)$. It suffices to show that f is continuous.

If f is not continuous, there exists a sequence $(w^k, t^k) = (w_1^k, \cdots, w_n^k, t^k)$ in $(D^*)^{n+1}$ such that $f(w^k, t^k) \to q \neq f(0, \cdots, 0)$. By applying Theorem 1 with $f_k(z) = f(zw^k/|w^k|, t^k)$ and $z_k' = |w^k|$, we see that $f(0, t^k) = f_k(0) \to q$. On the other hand, if we apply Theorem 1 with

$$f_k(z) = f(zt^k/|t^k|, \cdots, zt^k/|t^k|, t^k) \quad \text{and} \quad z_k' = |t^k|,$$

we see that $f(0, t^k) \to f(0, \cdots, 0) \neq q$. This is a contradiction and therefore f extends holomorphically. Q.E.D.

The following example shows that if M is not compact, restrictions on the singularities of A are necessary in order to make Theorem 2 remain true. Let $M = D \times (\mathbf{C} - \{-1, 1\}) \subset P_1(\mathbf{C}) \times P_1(\mathbf{C})$. Since both D and $\mathbf{C} - \{-1, 1\}$ are hyperbolically imbedded in $P_1(\mathbf{C})$, example (7) of § 2 implies that M is hyperbolically imbedded in $P_1(\mathbf{C}) \times P_1(\mathbf{C})$. Let $X = D \times D$ and $A = \{(z, w) | z = 0$ or $z = \pm w\}$ and define $f: X - A \to M$ by $f(z, w) = (z, w/z)$. Clearly f does not extend to all of X, since $f(0, 0)$ cannot be defined in a continuous way.

This example is more or less typical of the general situation. To see this consider the following generalization. Let A be a closed complex subspace of $X = D^n = D \times \cdots \times D$ with $(0, \cdots, 0) \in A$. Let L denote the tautological line bundle over

$P_{n-1}(\mathbf{C})$ (i.e. L is the line bundle associated to the principal $\mathbf{C}^*$ bundle $\pi: \mathbf{C}^n - \{(0, \cdots, 0)\} \to P_{n-1}(\mathbf{C})$ where π is the natural projection). Then $X - A$ can be considered as a subspace of L. If $X - A$ is hyperbolically imbedded in L, then the inclusion $i: X - A \to L$ is a counterexample to the singular version of Theorem 2. Note that in these examples the maps do extend to a meromorphic map. This is always the case.

Theorem 3. *Let A be a closed complex subspace of a complex space X and let M be hyperbolically imbedded in Y. If $f: X - A \to M$ is holomorphic, then f extends to a meromorphic map $f: X \to Y$.*

Proof. By a resolution of A in X we shall mean a triple (Z, B, ϕ) where

(i) Z is a nonsingular complex manifold (not necessarily connected).

(ii) B is a complex subspace of Z whose singularities are normal crossings.

(iii) $\phi: Z \to X$ is a proper holomorphic map onto X with $\phi^{-1}(A) = B$ and such that ϕ^{-1} is meromorphic. By Hironaka [2], such a resolution always exists (at least locally). Define $\bar{f}: Z - B \to M$ by $\bar{f} = f \circ \phi$. By Theorem 2, $\bar{f}$ extends to a holomorphic map $\bar{f}: Z \to Y$. Thus f extends meromorphically to X by defining $f = \bar{f} \circ \phi^{-1}$. Q.E.D.

We shall now weaken the restrictions on the singularities of A given in Theorem 2.

Theorem 4. *Let A be a closed complex subspace of a normal complex space X. Assume that (Z, B, ϕ) is a resolution of A in X (see proof above) such that for every pair of points p and q in B with $\phi(p) = \phi(q)$, there exist sequences $\{p_n\}$ and $\{q_n\}$ in $Z - B$ with $p_n \to p$, $q_n \to q$ and $d_{X-A}(\phi(p_n), \phi(q_n)) \to 0$. If M is hyperbolically imbedded in Y, then any holomorphic map $f: X - A \to M$ extends to a holomorphic map $f: X \to Y$.*

Proof. Let $\bar{f}: Z \to Y$ be defined as in the last proof. Since X is normal, it suffices to show that the meromorphic map $f = \bar{f} \circ \phi^{-1}$ is single valued. Since M is hyperbolically imbedded in Y and since $f|_{X-A}$ is distance decreasing, we have

$$\bar{f}(p) = \lim_{n\to\infty} f(\phi(p_n)) = \lim_{n\to\infty} f(\phi(q_n)) = \bar{f}(q).$$

This implies that f is single valued. Q.E.D.

Remark. If X is nonsingular and if A has only normal crossings, then A obviously satisfies the conditions of Theorem 4. Thus Theorem 4 is a generalization of Theorem 2, but in practice it is not a very satisfactory one because we are usually unable to actually construct a resolution. However, it does not seem very easy to give a condition which depends only on $X - A$ and which is not too restrictive.

In our final generalization of the Picard theorem, we consider the space $M_n = \{(w_1, \cdots, w_n) \in \mathbf{C}^n | w_i \neq 0$ and $w_i \neq 1$ for $i = 1, \cdots, n\}$. Let $[(w_1, v_1), \cdots, (w_n, v_n)]$ be "homogeneous" coordinates for $P_1(\mathbf{C}) \times \cdots \times P_1(\mathbf{C})$ and let $(w_0, w_1, \cdots, w_n)$ be homogeneous coordinates for $P_n(\mathbf{C})$. Define $\phi: \mathbf{C}^n \to P_1(\mathbf{C}) \times \cdots \times P_1(\mathbf{C})$ by $\phi(w_1, \cdots, w_n) = [(w_1, 1), \cdots, (w_n, 1)]$ and define $\psi: \mathbf{C}^n \to P_n(\mathbf{C})$ by $\psi(w_1, \cdots, w_n) = (1, w_1, \cdots, w_n)$. Thus ϕ imbeds M_n in $P_1(\mathbf{C}) \times \cdots \times P_1(\mathbf{C})$ and ψ imbeds M_n in $P_n(\mathbf{C})$. By (7) of § 2, M_n is hyperbolically imbedded in $P_1(\mathbf{C}) \times \cdots \times P_1(\mathbf{C})$ and therefore the results of this section apply in this case. On the other hand, an easy computation shows that the identity map $i: M_n \to M_n$ does not extend to a holomorphic map $i: P_1(\mathbf{C}) \times \cdots \times P_1(\mathbf{C}) \to P_n(\mathbf{C})$. Since the singularities of $A = P_1(\mathbf{C}) \times \cdots \times P_1(\mathbf{C}) - M_n$ are normal crossings, Theorem 2 implies that M_n is not hyperbolically imbedded in $P_n(\mathbf{C})$. However, we still have

Theorem 5. *Let A be a closed submanifold of a complex manifold X. Then every holomorphic map $f: X - A \to M_n$ extends to a holomorphic map $f: X \to P_n(\mathbf{C})$.*

Proof. By Theorem 2, f extends to a holomorphic map $f: X \to P_1(\mathbf{C}) \times \cdots \times P_1(\mathbf{C})$. We can assume $X = D^m = \{(z_1, \cdots, z_m) \in \mathbf{C}^m | |z_i| < 1$ for $i = 1, \cdots, m\}$ and that $A = \{(z_1, \cdots, z_m) \in D^m | z_1 = 0\}$. Let $f = (f_1, \cdots, f_n)$ where $f_i: X \to P_1(\mathbf{C})$. Without loss of generality, we can assume that f_i is given by two holomorphic functions, i.e. $f_i(z_1, \cdots, z_n) = (g_i(z_1, \cdots, z_n), h_i(z_1, \cdots, z_n))$. Since $h_i|_{X-A} \neq 0$, we see that either $h_i|_A \equiv 0$ or $h_i|_A \neq 0$. This implies that $h_i(z_1, \cdots, z_n) = z_1^{k_i} \hat{h}_i(z_1, \cdots, z_n)$ where $k_i \geq 0$ and $\hat{h}_i(z, \cdots, z_n) \neq 0$ in X. By the same reasoning, we have $g_i(z_1, \cdots, z_n) = z_1^{k_i'} \hat{g}_i(z_1, \cdots, z_n)$ where $\hat{g}_i(z_1, \cdots, z_n) \neq 0$ in X. This means that f (i.e. $\psi \circ f$) is given by

$$f(z_1, \cdots, z_n) = \left(1, z_1^{l_1} \frac{\hat{g}_1(z_1, \cdots, z_n)}{\hat{h}_1(z_1, \cdots, z_n)}, \cdots, z_1^{l_n} \frac{\hat{g}_n(z_1, \cdots, z_n)}{\hat{h}_n(z_1, \cdots, z_n)}\right)$$

where $l_i = k_i' - k_i$. This implies that f extends to a holomorphic map $f: X \to P_n(\mathbf{C})$. Q.E.D.

4. **Applications.** Let $\mathfrak{D}$ be a bounded symmetric domain in $\mathbf{C}^n$ and let Γ be an arithmetically defined, discrete subgroup of the largest connected group of holomorphic automorphisms of $\mathfrak{D}$. Assume Γ acts freely on $\mathfrak{D}$. There are two compactifications of $\mathfrak{D}/\Gamma$. The Satake compactification $(\mathfrak{D}/\Gamma)^*_{\mathrm{B.B.}}$ is the same as that of Borel-Baily as a topological space. The other compactification $(\mathfrak{D}/\Gamma)^*_{\mathrm{P.S.}}$ is that of Pyateckiĭ-Šapiro. In an unpublished work, Borel proved Corollary 1 for $(\mathfrak{D}/\Gamma)^*_{\mathrm{B.B.}}$. His proof shows that $\mathfrak{D}/\Gamma$ is hyperbolically imbedded in $(\mathfrak{D}/\Gamma)^*_{\mathrm{B.B.}}$. In [4], Kobayashi and Ochiai showed that $\mathfrak{D}/\Gamma$ is hyperbolically imbedded in $(\mathfrak{D}/\Gamma)^*_{\mathrm{P.S.}}$ and proved Corollary 1 for this case when A is nonsingular. Baily has shown that there is a one-one continuous map $\psi: (\mathfrak{D}/\Gamma)^*_{\mathrm{B.B.}} \to (\mathfrak{D}/\Gamma)^*_{\mathrm{P.S.}}$,

but it is not known whether $(\mathfrak{D}/\Gamma)^*_{\text{P.S.}}$ is even Hausdorff.(2) Using these remarks and letting M be either $(\mathfrak{D}/\Gamma)^*_{\text{B.B.}}$ or $(\mathfrak{D}/\Gamma)^*_{\text{P.S.}}$ we have

Corollary 1. *Let X and A be as in Theorem 4 (or 2). Then any holomorphic map $f: X - A \to \mathfrak{D}/\Gamma$ extends to a holomorphic map $f: X \to M$. If A is a complex subspace of a complex space X, then any holomorphic map $f: X - A \to \mathfrak{D}/\Gamma$ extends to a meromorphic map $f: X \to M$. In particular, if $f: \mathfrak{D}/\Gamma \to \mathfrak{D}'/\Gamma'$ is holomorphic, then f extends to a meromorphic map $f: M \to M'$.*

The next corollary can be interpreted as a "generalization" of Liouville's theorem.

Corollary 2. *Let X be a complex manifold with $d_X \equiv 0$ and let A be a closed complex subspace of X. If M is a compact hyperbolic space and $f: X - A \to M$ is holomorphic, then f is a constant map. (For example, any map from $\mathbf{C} - \{0, 1\}$ into a compact Riemann surface of genus ≥ 2 is a constant map.) If $X - A$ is hyperbolic, and Γ is a group of automorphisms of $X - A$ such that $X - A$ is a covering space of $(X - A)/\Gamma$, then $(X - A)/\Gamma$ is not compact.*

Proof. Since M is compact hyperbolic, f extends to a map $f: X \to M$. Since $d_X \equiv 0$ and d_M is a proper distance, the distance decreasing property of f implies that f is a constant map. Q.E.D.

Corollary 3. *Let $V \subset P_n(\mathbf{C})$ be a complex manifold and let $M_n \subset P_n(\mathbf{C})$ be as in Theorem 5. Let $M = V \cap M_n$. Let A be a closed complex subspace of a complex space X. Then any holomorphic map $f: X - A \to M$ extends to a meromorphic map $f: X \to V$. If X and A are both nonsingular, then $f: X \to V$ is holomorphic.*

Proof. Since M is hyperbolically imbedded in $P_1(\mathbf{C}) \times \cdots \times P_1(\mathbf{C})$, it is clear that f extends meromorphically. The second statement follows directly from Theorem 5. Q.E.D.

The previous corollary generalizes Corollary A of a recent paper of Griffiths [1]. Griffiths has shown that under the proper restrictions on the imbedding of V in $P_n(\mathbf{C})$, M will have the property that it is covered by a bounded domain in $\mathbf{C}^m$ which is topologically a cell.

5. **Conclusion.** In this paper we have considered generalizations of the big Picard theorem. Except for Theorem 5, we have replaced the range space $P_1(\mathbf{C}) - \{3 \text{ points}\}$ by a hyperbolically imbedded complex space, and then we have tried to replace the domain space D^* by more general spaces. Now we comment on the relevency of the assumption "hyperbolically imbedded".

To do this consider the following examples.

(i) Let $M' = \{(z_0, z_1, z_2) \in P_2(\mathbf{C}) \mid z_0 \neq 0,\ z_1 \neq 0,\ z_1 \neq z_0,\ z_2 \neq \pm z_0 e^{z_0/z_1}\}$.

(2) Borel has recently shown that the two compactifications of $\mathfrak{D}/\Gamma$ are equivalent.

Define $\phi: M' \to (\mathbf{C} - \{0, 1\}) \times (\mathbf{C} - \{-1, 1\})$ by

$$\phi(z_0, z_1, z_2) = (z_1/z_0, (z_2/z_0)e^{-z_0/z_1}).$$

Clearly ϕ is a biholomorphism and thus M' is complete hyperbolic. However, the map $g: D^* \to M'$ defined by $g(z) = (1, z, 2e^{1/z})$ does not extend to a meromorphic map $g: D \to P_2(\mathbf{C})$.

(ii) Let $M'' = \{(z_0, z_1, z_2) \in P_2(\mathbf{C}) | z_0 \neq 0, z_1 \neq 0, z_1 \neq z_0, z_2 z_1 \neq \pm z_0^2\}$. Since the map $h: D^* \times D \to M''$ defined by $h(z, w) = (1, z, w/z)$ does not extend to a holomorphic map $h: D \times D \to P_2(\mathbf{C})$, M'' is not hyperbolically imbedded in $P_2(\mathbf{C})$. Define $\psi: M'' \to (\mathbf{C} - \{0, 1\}) \times (\mathbf{C} - \{-1, 1\})$ by $\psi(z_0, z_1, z_2) = (z_1/z_0, z_1 z_2/z_0^2)$. It is easy to see that ψ is a biholomorphism which extends to a bimeromorphism $\psi: P_2(\mathbf{C}) \to P_1(\mathbf{C}) \times P_1(\mathbf{C})$. Thus Theorem 3 holds for maps into M''.

The first example shows that some assumptions on the way in which M is contained in Y are necessary if any extension theorem is going to work. The second example indicates that it is not necessary for M to be hyperbolically imbedded in Y in order for Theorem 3 to be true. In fact, it is possible that it may be sufficient for M to be a (complete) hyperbolic Zariski open set of a closed algebraic manifold Y. It also shows that these last conditions are not strong enough to imply that you can extend holomorphically over a nonsingular subspace. Theorem 5 implies that "hyperbolically imbedded" is not a necessary condition for the nonsingular version of Theorem 2. Finally, the discussion preceding Theorem 5 indicates that strong assumptions such as "hyperbolically imbedded" are probably necessary if Theorem 2 is to hold.

REFERENCES

1. P. A. Griffiths, *Complex-analytic properties of certain Zariski open sets on algebraic varieties*, Ann. of Math. (2) **94** (1971), 21-51.

2. H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*. I, II, Ann. of Math (2) 79 (1964), 109-326. MR 33 #7333.

3. S. Kobayashi, *Hyperbolic manifolds and holomorphic mappings*, Pure and Appl. Math., no. 2, Dekker, New York, 1970. MR 43 #3503.

4. S. Kobayashi and T. Ochiai, *Satake compactifications and the great Picard theorem*, J. Math. Soc. Japan 23 (1971), 340–350.

5. M. H. Kwack, *Generalization of the big Picard theorem*, Ann. of Math. (2) **90** (1969), 9–22. MR **39** #4445.

6. R. Remmert, *Holomorphe und meromorphe Abbildungen komplexer Räume*, Math. Ann. 133 (1957), 328–370. MR **19**, 1193.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER 8, B.C., CANADA